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Inclass Presentation #9

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3.31

First, we'll find the potential from the force:

$$f = kr^{-3} \Rightarrow V = \frac{k}{2r^2}$$

Goldstein (3.96) gives us:

$$\Theta(s) = \pi - 2\int_{r_m}^{\infty} \frac{s\,dr}{r\sqrt{r^2\left(1-\frac{V(r)}{E}\right)-s}}$$

$$= \pi - 2\int_{r_m}^{\infty} \frac{s\,dr}{r\sqrt{r^2 - s^2 - \frac{r^2 V(r)}{E}}}$$

$$= \pi - 2\int_{r_m}^{\infty} \frac{s\,dr}{r\sqrt{r^2 - s^2 - \frac{r^2 k}{2r^2 E}}} \quad \text{using } V = \frac{k}{2r^2}$$

$$= \pi - 2\int_{r_m}^{\infty} \frac{s\,dr}{r\sqrt{r^2 - s^2 - \frac{k}{2E}}}$$

Since we don't know r_m, we'll need to find an equation for it in terms of other parameters. The energy of the particle is:

$$E = \frac{1}{2}mr_m\dot{\theta}^2 + \frac{k}{2r_m^2}$$ when it passes closest to the center of force

Goldstein gives an equation for $\dot{\theta}$ when the particle is at point r_m on page 113:

$$\dot{\theta} = \frac{s}{r_m^2}\sqrt{\frac{2E}{m}} \Rightarrow \dot{\theta}^2 = \frac{s^2}{r_m^4}\frac{2E}{m}$$

Plugging this into our energy equation yields:

$$E = \frac{1}{2}mr_m^2\frac{s^2}{r_m^4}\frac{2E}{m} + \frac{k}{2r_m^2}$$

$\boxed{\text{3.31 CONTINUED}}$

$$E = \frac{s^2 E}{r_m^2} + \frac{k}{2 r_m^2}$$

$$E = \frac{1}{r_m^2}\left(s^2 E + \frac{k}{2}\right)$$

$$r_m^2 = s^2 + \frac{k}{2E}$$

We now recognize that these terms are in our original equation for $\Theta(s)$ which allows for the following substitution:

$$\Theta(s) = \pi - 2s \int_{r_m}^{\infty} \frac{dr}{r\sqrt{r^2 - r_m^2}}$$

H.B. Dwight's Integral Tables (281.08) gives me this integral as:

$$\Theta(s) = \pi - 2s\left[\frac{1}{r_m}\arccos\left(\frac{r_m}{r}\right)\right]_{r_m}^{\infty}$$

$$\Theta(s) = \pi - \frac{2s}{r_m}\left[\frac{\pi}{2} - 0\right]$$

$$\Theta(s) = \pi - \frac{\pi s}{r_m}$$

Now, we'll find $s(\Theta)$ because it will be useful in Goldstein (3.93)

$$\frac{\Theta}{\pi} = 1 - \frac{s}{r_m}$$

$$\frac{s}{r_m} = 1 - \frac{\Theta}{\pi}$$

$$s = r_m\left(1 - \frac{\Theta}{\pi}\right)$$

However, I know that r_m is a function of s, so we'll make use of my previous result for r_m^2

$$\Rightarrow s^2 = r_m^2\left(1 - \frac{\Theta}{\pi}\right)^2$$

$$\Rightarrow s^2 = \left(s^2 + \frac{k}{2E}\right)\left(1 - \frac{\Theta}{\pi}\right)^2$$

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$$s^2 = s^2\left(1-\frac{\Theta}{\pi}\right)^2 + \frac{k}{2E}\left(1-\frac{\Theta}{\pi}\right)^2$$

$$s^2\left(1-\left(1-\frac{\Theta}{\pi}\right)^2\right) = \frac{k}{2E}\left(1-\frac{\Theta}{\pi}\right)^2$$

$$s^2 = \frac{k}{2E}\frac{\left(1-\frac{\Theta}{\pi}\right)^2}{1-\left(1-\frac{\Theta}{\pi}\right)^2}$$

$$s = \sqrt{\frac{k}{2E}}\frac{1-\frac{\Theta}{\pi}}{\sqrt{1-\left(1-\frac{\Theta}{\pi}\right)^2}}$$

$$s = \sqrt{\frac{k}{2E}}\frac{1-\frac{\Theta}{\pi}}{\sqrt{1-\left(1-\frac{2\Theta}{\pi}+\frac{\Theta^2}{\pi^2}\right)}}$$

$$s = \sqrt{\frac{k}{2E}}\frac{1-\frac{\Theta}{\pi}}{\sqrt{\frac{2\Theta}{\pi}-\frac{\Theta^2}{\pi^2}}}$$

Using the substitution recommended by Goldstein of $x = \frac{\Theta}{\pi}$,

$$s = \sqrt{\frac{k}{2E}}\frac{1-x}{\sqrt{2x-x^2}}$$

Now, looking ahead to our ultimate goal of constructing Goldstein 3.93:

$$\sigma = \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right| ,$$

We'll now compute $\frac{ds}{d\Theta}$ noting that $dx = d\Theta\frac{1}{\pi}$.

$$\frac{ds}{d\Theta} = \frac{ds}{dx}\frac{dx}{d\Theta}$$

$$\frac{ds}{d\Theta} = \frac{1}{\pi}\left(\frac{ds}{dx}\right)$$

$\boxed{\text{3.31 CONTINUED}}$

Now, we'll find $\frac{ds}{dx}$:

$$\frac{ds}{dx} = \sqrt{\frac{k}{2E}}\;\frac{-\sqrt{2x-x^2} - \frac{(1-x)(2-2x)}{2\sqrt{2x-x^2}}}{2x-x^2}$$

$$\frac{ds}{dx} = \sqrt{\frac{k}{2E}}\;\frac{-\sqrt{2x-x^2} - \frac{(1-x)^2}{\sqrt{2x-x^2}}}{2x-x^2}$$

At this point, it will be beneficial to construct the entire σ equation to begin cancellation of some terms:

$$\sigma(\theta) = \frac{\sqrt{\frac{k}{2E}}\;\frac{1-x}{\sqrt{2x-x^2}}}{\sin(\pi x)}\sqrt{\frac{k}{2E}}\;\frac{-\sqrt{2x-x^2} - \frac{(1-x)^2}{\sqrt{2x-x^2}}}{2x-x^2}\;\frac{1}{\pi}$$

$$\sigma(\theta) = \frac{k}{2E}\;\frac{1}{\sin(\pi x)}\;\frac{1}{\pi}\underbrace{\left(\frac{1-x}{\sqrt{2x-x^2}}\;\frac{-\sqrt{2x-x^2} - \frac{(1-x)^2}{\sqrt{2x-x^2}}}{2x-x^2}\right)}_{\text{we will work on simplifying this alone for simplicity}}$$

$$\Rightarrow \frac{1-x}{\sqrt{2x-x^2}}\;\sqrt{2x-x^2}\;\frac{\left(-1-\frac{(1-x)^2}{2x-x^2}\right)}{2x-x^2}$$

$$(1-x)\;\frac{\left(-1-\frac{(1-x)^2}{2x-x^2}\right)}{2x-x^2}$$

$$(x-1)\;\frac{\left(1+\frac{(1-x)^2}{2x-x^2}\right)}{2x-x^2}$$

$$(x-1)\;\frac{2x-x^2+(1-x)^2}{(2x-x^2)^2}$$

$\boxed{\text{3.31 CONTINUED}}$

$$(x-1)\,\frac{2x-x^2+1-2x+x^2}{x^2(2-x)^2}$$

$$\frac{x-1}{x^2(2-x)^2}$$

However, this can be written as:

$$\frac{-(1-x)}{x^2(2-x)^2}$$

Since $x = \frac{\Theta}{\pi}$ and $\Theta \leq \pi$, x will always be less than one, and the above expression overall will be negative. We have been somewhat sloppy in dropping the absolute valve function in Goldstein 3.93, but it becomes relevant here in that it converts our final expression into

$$\frac{1-x}{x^2(2-x)^2}$$

Finally reconstructing $\sigma(\Theta)$,

$$\sigma(\Theta) = \frac{k}{2E}\,\frac{1}{\sin(\pi x)}\,\frac{1}{\pi}\,\frac{1-x}{x^2(2-x)^2}$$

Noting that $d\Theta = \pi\, dx$, we can arrive at Goldstein's solution:

$$\boxed{\sigma(\Theta)\, d\Theta = \frac{k}{2E}\,\frac{(1-x)\,dx}{x^2(2-x)^2\sin(\pi x)}}$$